QUANTIFIERS AND DIMENSION EQUIVALENCE RELATIONS ON ORTHOMODULAR LATTICES

By

JEAN HARWELL BEVIS

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CHAPTER I

INTRODUCTION

In this paper L will denote a complete orthomodular lattice. A quantifier on L is a mapping ϕ: $L \to L$, such that $0\phi = 0$, $e \leq e\phi$, and $(e \wedge f\phi) = e\phi \wedge f\phi$, where e and f are arbitrary elements of L. The pair (L,ϕ) is called a monadic lattice. We are concerned with the connection between quantifiers on L and equivalence relations on L which satisfy some or all of the following properties:

A) if $e \sim 0$, then $e = 0$,

B) finite divisibility, that is, if $e_1 \perp e_2$ and $f \sim e_1 \vee e_2$, then there exists an orthogonal decomposition of f, $f = f_1 \vee f_2$, such that $e_1 \sim f_1$ and $e_2 \sim f_2$,

B') complete divisibility, that is, if $\{e_\alpha\}$ is an orthogonal family and $f \sim \bigvee_\alpha e_\alpha$, then there exists an orthogonal family $\{f_\alpha\}$ over the same indexing set, such that $f = \bigvee_\alpha f_\alpha$ and $e_\alpha \sim f_\alpha$ for every α,

C) complete additivity, that is, if $\{e_\alpha\}$ is an orthogonal family and $\{f_\alpha\}$ is a second orthogonal family over the same indexing set, such that $e_\alpha \sim f_\alpha$ for every α, then $\bigvee_\alpha e_\alpha \sim \bigvee_\alpha f_\alpha$,

D) if e and f are not orthogonal, then there exists nonzero elements $e_1 \leq e$ and $f_1 \leq f$, such that $e_1 \sim f_1$,

D') if e and f have a common complement, then $e \sim f$.

A complete orthomodular lattice equipped with an equivalence relation satisfying axioms A, B, C, and D is called a Loomis dimension lattice.

If axioms A, B', C, and D' are satisfied, then the pair $(L,\sim)$ is called an orthomodular geometry. Sometimes the term dimension lattice is used instead of Loomis dimension lattice.

L. H. Loomis [9] has already investigated the properties of lattices equipped with such relations. We are interested in how these equivalence relations might arise in a natural way from other properties of the lattice. M. F. Janowitz [6] has shown that if $(L,\sim)$ is a Loomis dimension lattice satisfying B', and if L is type I with respect to $\sim$, then the relation is induced in a natural manner by *-equivalence in a suitable Baer *-semigroup which coordinatizes L. M. Donald Maclaren [11] has shown that if L is an M-symmetric, complete orthomodular lattice which is type I with respect to the central cover, then the central cover induces an equivalence relation on L satisfying A, B, C, and D'. However, this relation is not always the one which is naturally associated with the lattice. For example, if $\overline{L(H)}$ is the lattice of all closed subspaces of a Hilbert Space H, a natural dimension equivalence relation for $\overline{L(H)}$ is $M \sim N$ if and only if $\dim(M) = \dim(N)$. However, the equivalence relation constructed by M. Donald Maclaren has any two infinite dimensional closed subspaces equivalent. Thus, if $\dim(H)$ is more than countable, the two equivalence relations do not agree.

In a dimension lattice $(L,\sim)$ we say that an element, e, is invariant with respect to $\sim$, if e and e' contain no equivalent sub-elements. L. H. Loomis has shown that the set of all invariant elements forms a complete sublattice of the center of L. The question we are concerned with is, "Under what conditions does a complete sublattice of the center of a complete orthomodular lattice induce a dimension

equivalence relation on the lattice?" This is actually a question about center valued quantifiers, that is, quantifiers whose range is contained in the center of the lattice.

We conclude this chapter with some terminology and elementary properties concerning quantifiers on orthomodular lattices.

Definition 1.1. Let ϕ be a quantifier on the orthomodular lattice L,

(i) ϕ is called the discrete quantifier on L if it is the identity mapping of L onto L,

(ii) ϕ is called the indiscrete or simple quantifier on L if $e \neq 0$ implies $e\phi = 1$,

(iii) e is ϕ-invariant if $e\phi = e$,

(iv) e is ϕ-simple if $f \leq e$ implies $f = e \wedge f\phi$,

(v) e is ϕ-faithful if $e\phi = 1$,

(vi) e is type I with respect to ϕ if every subelement of e is an orthogonal supremum of ϕ-simple elements,

(vii) (L,ϕ) is type I if every element of L is type I with respect to ϕ.

Lemma 1.2. Let (L,ϕ) be a monadic lattice, then

(i) ϕ is an order endomorphism,

(ii) $1\phi = 1$,

(iii) $\phi = \phi^2$,

(iv) $(f\phi)'\phi = (f\phi)' \leq f'$ for all f in L,

(v) ϕ is a projection in S(L), where S(L) denotes the set of all residuated mappings on L, see [2],

(vi) ϕ is a complete join endomorphism,

(vii) the range of ϕ is a sub-complete sub-orthomodular lattice of L.

Proof: (i) $e \leq f$ implies that $e \leq f\phi$, which implies that $e \wedge f\phi = e$, that is, $(e \wedge f\phi)\phi = e\phi \wedge r\phi$, and thus $e\phi \leq f\phi$. (ii) Since $e \leq e\phi$ for all e in L, we have $1 \leq 1\phi$. Thus $1\phi = 1$. (iii) $(f\phi)\phi = (1 \wedge f\phi)\phi = 1\phi \wedge f\phi = f\phi$. (iv) $0 = 0\phi = [(f\phi)' \wedge f\phi]\phi = (f\phi)'\phi \wedge f\phi^2 = (f\phi)'\phi \wedge f\phi$. (v) By (i) and $0\phi = 0$, ϕ is monotone and normalized. From $e \leq e\phi$ and (iv) it follows that ϕ is a self-adjoint residuated mapping in S(L). Hence, by (iii), ϕ is a projection in S(L). (vi) Every member of S(L) is a complete join endomorphism. (vii) This is now clear.

Lemma 1.3. (i) A quantifier on an orthomodular lattice L determines and is determined by its invariant elements, that is, $\bigwedge\{f: f \text{ is } \phi\text{-invariant and } e \leq f\}$ exists and is equal to $e\phi$ for all e in L.

(ii) If M is a sub-complete sub-orthomodular lattice of the orthomodular lattice L, contained in the center of L, and if $\bigwedge\{f \in M: e \leq f\}$ exists for all e in L, then M determines a quantifier ϕ_M in the sense of (i) and, furthermore, M is the set of invariant elements of ϕ_M.

Proof. (i) Since $e \leq e\phi$, $e\phi \in \{f: f \text{ is } \phi\text{-invariant and } e \leq f\}$. Suppose that $f = g\phi$ and $e \leq f$, then $e\phi \leq f\phi = g\phi^2 = g\phi = f$. Therefore, $e\phi = \bigwedge\{f: f \text{ is } \phi\text{-invariant and } e \leq f\}$.

(ii) Now M is not empty, so that if e is in M, $e \wedge e' = 0$ is an element of M, thus, $0\phi_M = 0$. Clearly, $e \leq e\phi_M$ for all e in L. Note

that $e = e\phi_M$ if and only if e is in M, so that $\phi_M^2 = \phi_M$. If $e \leq f$ then $e \leq f\phi_M$ and hence $e\phi_M \leq f\phi_M$. Furthermore, $(e\phi_M)'$ is in M, so that $(e\phi_M)'\phi_M = (e\phi_M)' \leq e'$. Whence, ϕ_M is a self-adjoint idempotent in S(L) and, therefore, ϕ_M preserves suprema. Since $(f\phi_M)'$ is central, we have for any e and f in L $e = (e \wedge f\phi_M) \vee (e \wedge (f\phi_M)')$, and

$$e\phi_M = (e \wedge f\phi_M)\phi_M \vee (e \wedge (f\phi_M)')\phi_M \leq (e \wedge f\phi_M)\phi_M \vee (f\phi_M)'\phi_M$$
$$= (e \wedge f\phi_M)\phi_M \vee (f\phi_M)'.$$

Finally, $e\phi_M \wedge f\phi_M = [(e \wedge f\phi_M)\phi_M \vee (f\phi_M)'] \wedge f\phi_M = (e \wedge f\phi_M)\phi_M$.

Definition 1.4. For e and f in L, we say that e is ϕ-unrelated to f in case $e\phi$ is orthogonal to $f\phi$. Otherwise, we say that e and f are ϕ-related. Sometimes the term very orthogonal is used instead of unrelated. f is a ϕ-descendant of e in case $f \leq e\phi$.

Lemma 1.5. If ϕ is a center valued quantifier on a complete orthomodular lattice L, then

(i) $e\phi$ is the smallest invariant element including e,

(ii) $(e\phi)'$ is the maximum element unrelated to e,

(iii) f is unrelated to e if and only if $f \leq (e\phi)'$,

(iv) $e\phi$ is the maximum descendent of e,

(v) a descendant of a descendant of e is a descendant of e,

(vi) if f is unrelated to e, then f is unrelated to every descendant of e.

Proof: These properties are immediate consequences of the definitions.

CHAPTER II

MULTIPLICITY FAMILIES

For the remainder of this paper we shall be concerned with center valued quantifiers, ϕ, on L. Where no confusion may arise we shall write $|e|$ instead of $e\phi$. $|e|$ is called "the hull of e." Thus, $(L,|\ |)$ will denote a monadic lattice with a center valued quantifier. Furthermore, when there is only one quantifier under discussion, no confusion will arise from using the terms simple, invariant, etc., instead of ϕ-simple, ϕ-invariant, etc.

Lemma 2.1. A subelement of a simple element of $(L,|\ |)$ is simple.

Proof: Suppose that e is simple and that $b \leq a \leq e$.

Now $b \vee (a \wedge |b|') \leq a$ and $|b \vee (a \wedge |b|')| = |b| \vee (|a| \wedge |b|') = |a|$. Thus, since e is simple and $b \vee (a \wedge |b|') \leq e$, we have $b \vee (a \wedge |b|') = |b \vee (a \wedge |b|')| \wedge e = |a| \wedge e = a$.

Hence, $a \wedge |b| = [b \vee (a \wedge |b|')] \wedge |b| = b$ for every $b \leq a$. Therefore, a is simple.

Lemma 2.2. If e is related to $\bigvee_\alpha e_\alpha$, then e is related to some e_α.

Proof: If not, $|e| \leq |e_\alpha|'$ for every α, so $|e| \leq \bigwedge_\alpha |e_\alpha|' = (\bigvee_\alpha |e_\alpha|)' = |\bigvee_\alpha e_\alpha|'$. Thus e is unrelated to $\bigvee_\alpha e_\alpha$, a contradiction.

Lemma 2.3. If $\{e_\alpha\}$ is a family of mutually unrelated elements, and if $e = \bigvee_\alpha e_\alpha$, then $e_\alpha = |e_\alpha| \wedge e$ for every α.

Proof: For $\alpha \neq \beta$ we have $e_\alpha \leq |e_\alpha| \leq |e_\beta|' \leq e_\beta'$.

Now, $e = e_\alpha \vee \bigvee_{\alpha\neq\beta} e_\beta$ so that $e \wedge |e_\alpha|$

$= |e_\alpha| \wedge (e_\alpha \vee \bigvee_{\alpha\neq\beta} e_\beta) = e_\alpha \vee (|e_\alpha| \wedge \bigvee_{\alpha\neq\beta} e_\beta) = e_\alpha$,

since $\bigvee_{\alpha\neq\beta} e_\beta \perp |e_\alpha|$.

Lemma 2.4. If $\{e_\alpha\}$ is a family of mutually unrelated simple elements, then $e = \bigvee_\alpha e_\alpha$ is a simple element.

Proof: Suppose $a \leq e$. By the previous lemma

$e_\alpha \wedge a = |e_\alpha| \wedge e \wedge a = |e_\alpha| \wedge a$. Since e_α is simple

$e_\alpha \wedge a = e_\alpha \wedge |e_\alpha \wedge a| = e_\alpha \wedge \big||e_\alpha| \wedge a\big| = e_\alpha \wedge |e_\alpha| \wedge |a| = e_\alpha \wedge |a|$.

Hence, $e_\alpha \wedge |a| = e_\alpha \wedge a = |e_\alpha| \wedge a$. Thus,

$e \wedge |a| = \bigvee_\alpha e_\alpha \wedge |a| = \bigvee_\alpha (e_\alpha \wedge |a|) = \bigvee_\alpha (|e_\alpha| \wedge a) = \bigvee_\alpha |e_\alpha| \wedge a$

$= |e| \wedge a = a$.

Lemma 2.5. If $(L, |\ |)$ is type I and $e \in L$, then e contains a simple element f, such that $|e| = |f|$.

Proof: Let $\{f_\alpha\}$ be a maximal family of unrelated simple subelements of e, and let $f = \bigvee_\alpha f_\alpha$. Now f is simple by the previous lemma, and $f \leq e$, so $|f| \leq |e|$. If $e \wedge |f|' = 0$, then (since f is central) $e \leq |f|$ and $|e| \leq |f|$, thus $|e| = |f|$. On the other hand, if $e \wedge |f|' \neq 0$, there exists a nonzero simple subelement $c \in L$, such that $c \leq e \wedge |f|' \leq |f|' \leq \bigvee_\alpha |f_\alpha|'$. Since c is nonzero, simple, and unrelated to every f_α, we have a contradiction to the maximality of $\{f_\alpha\}$.

Definition 2.6. An element of type I is said to be type I_F, or finite, in case it does not contain an orthogonal sequence $\{e_i\}$ of nonzero simple subelements with $|e_i| = |e_j|$, $i,j = 1,2,\ldots$.

It is clear that a subelement of a finite element is finite. For the remainder of this paper we shall assume that $(L,|\,|)$ is type I.

It is possible to develop the theory of multiplicity families without considering the finite elements by themselves. However, the process is enlightening and augments the understanding of the general case.

Lemma 2.7. Let a be a finite element of L. Using lemma 2.5, we construct a sequence of orthogonal simple elements $\{a_1, a_2, a_3, \ldots\}$ such that

$$a_1 \leq a, \qquad |a_1| = |a|$$

$$a_2 \leq a \wedge a_1' \qquad |a_2| = |a \wedge a_1'|$$

$$\vdots$$

$$a_n \leq a \wedge a_1' \wedge \ldots \wedge a_{n-1}', \quad |a_n| = |a \wedge a_1' \wedge \ldots \wedge a_{n-1}'|$$

$$\vdots$$

Then, $\bigwedge_n |a \wedge a_1' \wedge \ldots \wedge a_n'| = 0$, and $a = \bigvee_n a_n$.

Proof: Let $e = \bigwedge_n |a \wedge a_1' \wedge \ldots \wedge a_n'|$, $c_n = e \wedge a_n$. The elements $c_1, c_2, \ldots$ are mutually orthogonal and simple. Moreover, $|c_n| = |e \wedge a_n| = e \wedge |a_n| = e \wedge |a \wedge a_1' \wedge \ldots \wedge a_{n-1}'| = e$. Since a is of type I_F, it follows that every c_n must be 0, hence $e = 0$. Finally,

$a \wedge \bigwedge_n a_n' = \bigwedge_n (a \wedge a_1' \wedge \ldots \wedge a_n') \leq \bigwedge_n |a \wedge a_1' \wedge \ldots \wedge a_n'| = 0$, hence $a \wedge \bigwedge_n a_n' = 0$. It follows that $a = \bigvee_n a_n$.

Lemma 2.8. Let $e \in L$, and let b be simple. If bCe (b commutes with e) and $|e \wedge a| \geq |e \wedge b|$, then $|e \wedge a'| \leq |e \wedge b'|$.

Proof: Suppose $|e \wedge a'| \not\leq |e \wedge b'|$. Since $|e \wedge a'|$ and $|e \wedge b'|$ are central, we have $|e \wedge a'| \wedge |e \wedge b'|' = f \neq 0$. Note that f is central. Now, $|e \wedge b'| \wedge f = |e \wedge b'| \wedge f = 0$, so $e \wedge b' \wedge f = 0$. Since bCe we have $e \wedge f \leq b$. Thus, $e \wedge a' \wedge f \leq b$, so $b_1 = (b \wedge f') \vee (e \wedge a' \wedge f) \leq b$. But, $|b_1| = |b \wedge f'| \vee |e \wedge a' \wedge f| = (|b| \wedge f') \vee f = |b| \vee f \geq b$. Since b is simple, $b_1 = b = (b \wedge f') \vee (e \wedge a' \wedge f)$. Since $e \wedge f \leq b$, then $e \wedge a \wedge f \leq b \wedge f = e \wedge a' \wedge f \leq a'$, hence $e \wedge a \wedge f = 0$. But, $0 = |e \wedge a \wedge f| = |e \wedge a| \wedge f \geq |e \wedge b| \wedge f = |e \wedge b \wedge f| = 0$, so $e \wedge b \wedge f = e \wedge a' \wedge f = 0$. But, $f = |e \wedge a' \wedge f|$, so $f = 0$, a contradiction.

Corollary 2.9: If a and b are simple, if a and b commute with e, and if $|e \wedge a| = |e \wedge b|$, then $|e \wedge a'| = |e \wedge b'|$.

Lemma 2.10. b is simple if and only if orthogonal subelements of b have orthogonal hulls.

Proof: Let b be simple, $a, c \leq b$, $a \perp c$, $g = |a| \wedge |c| \leq |b|$. Now, $|g \wedge a| = g$ and $|g \wedge c| = g$, so $|g \wedge a| \geq g \wedge b$ and $|g \wedge c| \geq g \wedge b$. Since $g \wedge b$ is simple, $g \wedge a = g \wedge b = g \wedge c \leq a \wedge c = 0$. Thus $g = g \wedge |b| = |g \wedge b| = |0| = 0$. Conversely, suppose that orthogonal subelements of b have orthogonal hulls. If b is not simple, there exists $a < b$ such that $b \not\leq |a|$. Thus $|b| = |a|$. But, $b = a \vee (b \wedge a')$, so $|b| = |a| \vee |b \wedge a'| = |a|$. Since $|a| \perp |b \wedge a'|$, $|b \wedge a'| = 0$, so $b \wedge a' = 0$, a contradiction.

Definition 2.11. Let $a \in L$, construct, as in lemma 2.7, a sequence, $\{a_1, a_2, \ldots\}$, of orthogonal simple subelements of a. Define the sequence $\{u_n\}$, called a multiplicity sequence for a, by $u_n = |a_n| \wedge |a_{n+1}|'$.

The multiplicity sequence of an element of type I_F of a complete orthomodular lattice with a center valued quantifier is in general not unique. For example, if L is the lattice whose Hasse diagram is given in Figure 1 and ϕ is the simple quantifier on L, then the unit element of (L,ϕ) has two multiplicity sequences, namely $\{0, 1, 0, 0, \ldots\}$ and

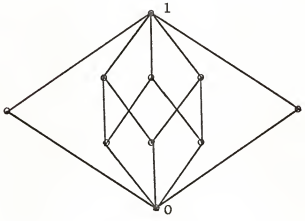

Figure 1

$\{0, 0, 1, 0, 0, \ldots\}$. From L. H. Loomis' work, [9], it is clear that the multiplicity sequences must be unique if ϕ is to induce a dimension equivalence relation. Lemma 2.13, below, leads us to a reasonable condition to impose on (L,ϕ) so that the multiplicity sequences will be unique.

Lemma 2.12. Let a be a finite element of L, and let $\{a_n\}$ and $\{u_n\}$ be as in definition 2.11. Then

(i) $|a_n| = |a_{n+1}| \vee u_n$ and $|a_{n+1}|$ is orthogonal to u_n,

(ii) $a_1 \wedge u_n$, $a_2 \wedge u_n$, . . . , $a_n \wedge u_n$ are orthogonal simple subelements of a with common hull u_n,

(iii) $a \wedge u_n = (a_1 \wedge u_n) \vee (a_2 \wedge u_n) \vee \ldots \vee (a_n \wedge u_n)$,

(iv) $a_{n+1} \wedge u_n = a_{n+2} \wedge u_n = \ldots = 0$,

(v) $\bigvee_n u_n = |a|$,

(vi) the elements $\{u_n\}$ are mutually orthogonal.

Proof: (i) follows immediately from definition 2.11 and the fact that $|a_n| \geq |a_{n+1}|$. For (ii), let $k \leq n$. Then,

$|a_k \wedge u_n| = |a_k| \wedge |a_n| \wedge |a_{n+1}|' = |a_n| \wedge |a_{n+1}|' = u_n$.

In (iii), that $(a_1 \vee a_2 \vee \ldots \vee a_n) \wedge u_n \leq a \wedge u_n$ is clear. But, $|a \wedge u_n \wedge [(a_1' \wedge a_2' \wedge \ldots \wedge a_n') \vee u_n']| = |a \wedge a_1' \wedge \ldots \wedge a_n'| \wedge u_n$

$= |a_{n+1}| \wedge u_n = 0$, so

$(a_1 \vee a_2 \vee \ldots \vee a_n) \wedge u_n = (a_1 \wedge u_n) \vee (a_2 \wedge u_n) \vee \ldots \vee (a_n \wedge u_n)$

$= a \wedge u_n$. In (iv), let $k > n$. Then, $|a_k \wedge u_n| = |a_k| \wedge |a_n| \wedge |a_{n+1}|'$

$= |a_k| \wedge |a_{n+1}|'$. Since $|a_{n+1}| \geq |a_k|$, we have $|a_k| \wedge |a_{n+1}|' = 0$.

For (v), let $s = \bigvee_{n=1}^{\infty} u_n \leq |a|$. Now, $s \geq \bigvee_{j=1}^{n} u_j = |a_1| \wedge |a_{n+1}|'$,

so $s' \leq |a_{n+1}| \vee |a|'$, $s' \wedge |a| \leq |a_{n+1}|$, and $s' \wedge |a| \leq \bigwedge_{n=1}^{\infty} |a_n| = 0$

by lemma 2.7. Thus, $s = |a|$. Finally, (vi) is clear since $|a| = |a_1| \geq |a_2| \geq \ldots$.

Theorem 2.13. Let a be a finite element of L. Let $\{u_n\}$ be a sequence of invariant elements such that:

(i) $\{u_n\}$ is an orthogonal sequence,

(ii) $\bigvee_n u_n = |a|$,

(iii) $a \wedge u_n = a_1^n \vee a_2^n \vee \ldots \vee a_n^n$ where $\{a_j^n : j=1,2, \ldots ,n\}$ is a collection of orthogonal simple elements with a common hull.

Let $a_n = a_n^n \vee a_n^{n+1} \vee \ldots = \bigvee_{j=n}^{\infty} a_n^j$. Then, the elements $a_1, a_2, \ldots$ are simple and orthogonal, $a = \bigvee_n a_n$, $|a_1| = |a|$, $|a_n| = |a_1 \wedge a_2' \wedge \ldots \wedge a_{n-1}'|$, and $u_n = |a_n| \wedge |a_{n+1}|'$. Thus, $\{u_n\}$ is a multiplicity sequence for a.

Proof: $u_n = |a| \wedge u_n = |a \wedge u_n| = |a_1^n| \vee |a_2^n| \vee \ldots \vee |a_n^n| = |a_1^n| = |a_2^n| = \ldots = |a_n^n|$ by (iii). Also, $|a_n^{n+p}| = u_{n+p}$, so $|a_n^{n+p}| \perp |a_n^{n+q}|$ for $p \neq q$, $p,q = 1, 2, \ldots$ $n = 1, 2, \ldots$. Thus, for $p \neq q$, a_n^{n+p} and a_n^{n+q} are unrelated simple elements. Thus, by lemma 2.4, a_n is a simple element for $n = 1, 2, \ldots$. Hence, $\{a_n\}$ is a collection of orthogonal simple elements. Now,

$$|a_1| = |a_1^1| \vee |a_1^2| \vee \ldots = u_1 \vee u_2 \vee \ldots = |a|$$

$$|a_2| = |a_2^2| \vee |a_2^3| \vee \ldots = u_2 \vee u_3 \vee \ldots$$

$$\vdots$$

$$|a_n| = |a_n^n| \vee |a_n^{n+1}| \vee \ldots = u_n \vee u_{n+1} \vee \ldots .$$

Also, $a = a \wedge |a| = a \wedge \bigvee_n u_n = \bigvee_n (a \wedge u_n)$

$= a_1^1 \vee (a_1^2 \vee a_2^2) \vee (a_1^3 \vee a_2^3 \vee a_3^3) \vee \ldots \vee (a_1^n \vee a_2^n \vee \ldots \vee a_n^n) \vee \ldots$

$= (a_1^1 \vee a_1^2 \vee a_1^3 \vee \ldots) \vee (a_2^2 \vee a_2^3 \vee a_2^4 \vee \ldots) \vee \ldots = a_1 \vee a_2 \vee a_3 \vee \ldots$. Finally, $a \wedge a_1' \wedge a_2' \wedge \ldots \wedge a_{n-1}' = a_n \vee a_{n+1} \vee \ldots$ and, therefore,

$|a \wedge a'_1 \wedge a'_2 \wedge \ldots \wedge a'_{n-1}| = |a_n| \vee |a_{n+1}| \vee \ldots = u_n \vee u_{n+1} \vee \ldots$

$= |a_n|$, which concludes the proof.

Suppose that a is an element of type I_F and that it has a unique multiplicity sequence. If u is an element of this sequence then by the previous theorem $u \wedge a$ is the join of a number of nonzero, orthogonal, simple elements with common hull. The number of these elements determines the position of u in the multiplicity sequence of a. Since the multiplicity sequence is unique, the cardinality of any collection of nonzero, orthogonal, simple elements with a common hull which has $u \wedge a$ as its union, must be the same. We now formalize this condition.

Definition 2.14. If a is an element of $(L,|\ |)$, not necessarily finite, then a is said to have uniform multiplicity c, provided that c is the smallest cardinal number such that $a \wedge e$ can be expressed as the union of c, nonzero, orthogonal, simple elements, where e is any invariant element such that $a \wedge e \neq 0$. An element a is said to have uniform multiplicity if there is some cardinal number c, such that a has uniform multiplicity c.

Definition 2.15. $(L,|\ |)$ is said to be piecewise semimodular if for every element a, of uniform multiplicity, there is some cardinal number c, such that, if $\{a_\alpha\}$ is any family of nonzero, orthogonal, simple subelements of a with $|a_\alpha| = |a|$, then the cardinality of the family $\{a_\alpha\}$ is c.

Lemma 2.16. If $(L,|\ |)$ is piecewise semimodular, then the multiplicity sequence of each finite element is unique.

Proof: Let $\{u_n\},\{v_m\}$ be two multiplicity sequences for a finite

element a. Now $|a| = \bigvee_n u_n = \bigvee_m v_m$, thus, $u_n = u_n \wedge \bigvee_m v_m = \bigvee_m (u_n \wedge v_m)$ for all n. Hence, for each $u_n \neq 0$ there is at least one v_m such that $u_n \wedge v_m \neq 0$. Suppose there were a v_m, $m \neq n$, such that $u_n \wedge v_m \neq 0$. Then $a \wedge u_n = \bigvee_{i=1}^{n} a_i$ and $a \wedge v_m = \bigvee_{j=1}^{m} b_j$, where $\{a_i\}$ and $\{b_j\}$ are families of nonzero, orthogonal, simple elements with $|a_i| = u_n$, $i = 1, 2, \ldots, n$ and $|b_j| = v_m$, $j = 1, 2, \ldots, m$. Furthermore, $\bigvee_{i=1}^{n}(a_i \wedge v_m) = \bigvee_{i=1}^{n} a_i \wedge v_m = a \wedge u_n \wedge v_m = u_n \wedge \bigvee_{j=1}^{m} b_j = \bigvee_{j=1}^{m}(b_j \wedge u_n)$, also, $|a_i \wedge v_m| = |a_i| \wedge v_m = u_n \wedge v_m$ and $|b_j \wedge u_n| = |b_j| \wedge u_n = v_m \wedge u_n$. Hence, $a \wedge u_n \wedge v_m$ can be written as the union of m or n ($m \neq n$) nonzero, orthogonal, simple elements with the common hull $u_n \wedge v_m = |a \wedge u_n \wedge v_m|$, contradicting piecewise semimodularity. Thus, for each $u_n \neq 0$, $u_n \wedge v_n \neq 0$ and $u_n \wedge v_m = 0$ if $m \neq n$. We have therefore deduced $u_n = u_n \wedge \bigvee_m v_m = \bigvee_m (u_n \wedge v_m) = u_n \wedge v_n$, for each $u_n \neq 0$. If we had started with the identity $v_m = v_m \wedge \bigvee_n u_n = \bigvee_n (v_m \wedge u_n)$ and used an argument similar to the one above, we would have obtained the identity $v_m = u_m \wedge v_m$ for all $v_m \neq 0$. It thus follows that $u_n = 0$ if and only if $v_n = 0$ and, hence, $u_n = u_n \wedge v_n = v_n$ for all n.

Definition 2.17. If $(L, |\ |)$ is piecewise semimodular, then two finite elements are said to be equivalent, in symbols $a \sim b$, if they have the same multiplicity sequence.

Hence, we have the following corollary to theorem 2.13.

Corollary 2.18. If $(L, |\ |)$ is piecewise semimodular, and if a is a finite element of $(L, |\ |)$, then the multiplicity sequence for a is uniquely determined by the properties:

(i) $\{u_n\}$ is an orthogonal sequence of invariant elements,

(ii) $\bigvee_n u_n = |a|$,

(iii) on u_n, a is the orthogonal union of n, equivalent, simple elements.

Lemma 2.19. Let $(L,|\ |)$ be piecewise semimodular, and let a be a finite element of $(L,|\ |)$ with uniform multiplicity N, N a finite cardinal number, then a is the union of N, orthogonal simple elements $\{a_j\}$ such that $|a| = |a_j|$, $j = 1, 2, \ldots, N$.

Proof: Let $\{u_n\}$ be the multiplicity sequence for a, so that $a = a \wedge |a| = a \wedge \bigvee_n u_n = \bigvee_n (a \wedge u_n)$. Now, if $n \prec N$, then $u_n \wedge a = 0$, otherwise we contradict the uniform multiplicity of a. Suppose that $n \geq N$, and $u_n \wedge a \neq 0$. Thus, $u_n \wedge a = \bigvee_{j=1}^{N} a_j$, where $\{a_j: j=1,\ldots,N\}$ are nonzero, orthogonal, simple elements, and $u_n \wedge a$ cannot be written as the union of fewer than N such elements. If $|a_1| \wedge |a_2| \wedge \ldots \wedge |a_N| = 0$, then $|a_1| \leq (|a_2| \wedge \ldots \wedge |a_N|)' = |a_2|' \vee \ldots \vee |a_N|'$, that is $a_1 = a_1 \wedge |a_1| \leq a_1 \wedge (|a_2|' \vee \ldots \vee |a_N|') = (|a_2|' \wedge a_1) \vee \ldots \vee (|a_N|' \wedge a_1)$. Therefore, $a \wedge u_n = \bigvee_{j=2}^{N} [a_j \vee (a_1 \wedge |a_j|')]$ is the join of N-1, nonzero orthogonal, simple elements. This contradiction implies that $e = |a_1| \wedge |a_2| \wedge \ldots \wedge |a_N| \neq 0$. By theorem 2.13, $a \wedge u_n \wedge e = \bigvee_{i=1}^{n} b_i \wedge e = \bigvee_{i=1}^{n} (b_i \wedge e)$, where $\{b_i\}$ is a collection of nonzero, orthogonal, simple elements with u_n as their common hull. Hence, $\{b_i \wedge e\}$ is a collection of n, nonzero, orthogonal, simple elements with e as their common hull. Furthermore, $a \wedge u_n \wedge e = \bigvee_{j=1}^{N} a_j \wedge e = \bigvee_{j=1}^{N} (a_j \wedge e)$ where $\{a_j \wedge e\}$ is a collection of N, nonzero, orthogonal, simple elements with e as their common hull. By piecewise semimodularity, $n = N$. Hence, $a = a \wedge u_N$. The result now follows from theorem 2.13 (iii).

Later on we shall see reason for naming this condition as we have. Now, we generalize the concept of multiplicity sequence to an arbitrary element of $(L, |\ |)$.

Definition 2.20. An element a of $(L, |\ |)$ is purely infinite if $a \wedge e$ is 0 or not finite for all invariant elements e.

Lemma 2.21. Every element a has a unique decomposition $a = a_1 \vee a_2$, into unrelated elements such that a_1 is finite and a_2 is purely infinite.

Proof: Let $\{e_\alpha\}$ be the set of all invariant elements such that $e_\alpha \leq |a|$, $e_\alpha \wedge a$ is finite, and $e_\alpha \wedge a \neq 0$. Let $e = \bigvee_\alpha e_\alpha$. We claim that $a \wedge e$ is finite. If not, there exists an orthogonal family $\{f_i: i = 1, 2, \ldots\}$, where each f_i is simple, $f_i \leq a \wedge e$, and $|f_i| = |f_j|$ for all i, j. Hence, $e_\alpha \wedge f_i \leq e_\alpha \wedge e \wedge a = e_\alpha \wedge a$, and $\{e_\alpha \wedge f_i: i = 1, 2, \ldots\}$ is an orthogonal set of simple elements. Furthermore, $|e_\alpha \wedge f_i| = e_\alpha \wedge |f_i| = e_\alpha \wedge |f_j| = |e_\alpha \wedge f_j|$. Since $e_\alpha \wedge a$ is finite for all α, $e_\alpha \wedge f_i = 0$ for all i and for all α. Thus, $f_i = f_i \wedge e = f_i \wedge \bigvee_\alpha e_\alpha = \bigvee_\alpha (f_i \wedge e_\alpha) = 0$ for all i. Hence, $a \wedge e$ is finite and e is the maximal such invariant element. Now, $a = (a \wedge e) \vee (a \wedge e')$, where $a \wedge e$ is finite and $a \wedge e'$ must be purely infinite by the construction of e. As to uniqueness, suppose that $a = b_1 \vee b_2$ is another such decomposition of a. Since $a \wedge |b_1|$ is finite, we have $|b_1| \leq e$. Now $a \wedge e \wedge |b_1|'$ is a finite subelement of $(b_1 \vee b_2) \wedge (e \wedge |b_1|') = b_2 \wedge e \wedge |b_1|'$. Therefore, $a \wedge e \wedge |b_1|' = 0$ and, thus, $0 = |0| = |a| \wedge e \wedge |b_1|' = e \wedge |b_1|'$. Hence, $e = |b_1|$. From this it easily follows that $a_1 = b_1$ and $a_2 = b_2$.

Lemma 2.22. If $\{e_\alpha\}$ is an orthogonal family of simple elements

and e_1 is some element of that family, then there exists a simple element f, such that $f \leq \bigvee_\alpha e_\alpha$, $e_1 \leq f$, $|f| = |\bigvee_\alpha e_\alpha|$, and f is the union of a family of subelements of the family $\{e_\alpha\}$, thus, fCe_α for all α.

Proof: Let $a = \bigvee_\alpha e_\alpha$, and suppose that $\{e_\alpha\}$ has been well ordered with e_1 as the first element of that ordering. If $|e_1| = |a|$ let $f = e_1$, otherwise $|e_1| \prec |a|$, which implies that there must be a nonzero element in the set $\{e_2 \wedge |e_1|', e_3 \wedge |e_1|', \ldots, e_\alpha \wedge |e_1|', \ldots\}$. Let E_2 be the first nonzero element of this set. Now e_1 and E_2 are mutually unrelated simple elements, so that $e_1 \vee E_2$ is a simple element. Furthermore, $|e_1 \vee e_2| \leq |e_1 \vee E_2| \leq a$. Continuing by induction, we suppose that E_γ has been chosen for all $\gamma \prec \delta$ (δ a fixed ordinal) such that $\{e_1, E_2, E_3, \ldots, E_\gamma, \ldots : \gamma \prec \delta\}$ is a family of mutually unrelated subelements of the family $\{e_\alpha\}$, and such that $|\bigvee_{1 \leq \gamma \prec \delta} e_\gamma| \leq |e_1 \vee \bigvee_{2 \leq \gamma \prec \delta} E_\gamma| \leq |a|$. It follows that $h = e_1 \vee \bigvee_{2 \leq \gamma \prec \delta} E_\gamma$ is simple. If $|h| \prec |a|$ then there is a first nonzero element E_δ of the set $\{e_\delta \wedge |h|', e_{\delta+1} \wedge |h|', \ldots, e_\alpha \wedge |h|', \ldots : \delta \leq \alpha\}$. Now, $\{e_1, E_2, E_3, \ldots, E_\gamma, \ldots : \gamma \leq \delta\}$ is a collection of mutually unrelated subelements of the set $\{e_\alpha\}$, such that $|\bigvee_{1 \leq \gamma \leq \delta} e_\gamma| \leq |h \vee e_\delta| \leq |a|$. The induction ends when we have a family $\{e_1, E_2, \ldots, E_\gamma, \ldots\}$ such that the hull of the union of this family is the same as the hull of a. Let f be the union of this family, thus, $|f| = |a|$. Furthermore, since f is the union of mutually unrelated subelements of the set $\{e_\alpha\}$, it is clear that f is simple by lemma 2.4, and that $f \leq \bigvee_\alpha e_\alpha$. Since

$E_\gamma Ce_\alpha$ for all γ and α, and $e_1 Ce_\alpha$ for all α, it follows that fCe_α for all α.

Lemma 2.23. If $\{f_\alpha: \alpha \in A\}$ and $\{e_\alpha: \alpha \in A\}$ are two collections of infinite cardinality c, well ordered by the same indexing set A, of orthogonal, simple elements of $(L,|\ |)$, such that $|f_\alpha| = u$ for all α and $\alpha \leq \beta$ implies $|e_\beta| \leq |e_\alpha| \leq u$, then $\bigvee_\alpha f_\alpha \vee \bigvee_\alpha e_\alpha$ is the union of c, orthogonal, simple elements each of which has u as its hull.

Proof: Let p be the sequence defined on the nonnegative integers, whose beginning terms are 0, 0, 1, 0, 2, 1, 3, 0, 4, 2, 5, 1, 6, 3, 7, . . . , and which is formed by certain blocks of length 2^n, n = 0, 1, 2, ... These blocks are ordered by their length so that the shorter blocks come first. In the n^{th} block, which has length 2^n, each of the positive integers 0, 1, 2, . . . , 2^n, appears once and only once in the following order: the first term is 0, the 2^{n-1} term is 1, the 2^{n-2} term is 2, the $2^{n-2}+2^{n-1}$ term is 3, the 2^{n-3} term is 4, the $2^{n-3}+2^{n-2}$ term is 5, the $2^{n-3}+2^{n-1}$ term is 6, the $2^{n-3}+2^{n-2}+3^{n-1}$ term is 7, etc. Let q be the sequence defined on the nonnegative integers as follows:

$$q(n) = \begin{cases} k & \text{if } n = 2k \\ p(k) & \text{if } n = 2k\ 1. \end{cases}$$

Now for each limit ordinal λ in A, we define two collections

$G_\lambda = \{g_{\lambda+n} = e_{\lambda+n} \vee (f_{\lambda+2n} \wedge |e_{\lambda+n}|'): \ n = 0, 1, 2, \ldots\}$ and

$H_\lambda = \{h_{\lambda+n} = (f_{\lambda+n} \wedge |e_{\lambda+q(n)}|) \vee (f_{\lambda+2n+1} \wedge |e_{\lambda+q(n)}|'): n = 0, 1, 2, \ldots\}$.

One may check to see that

$$\bigvee_{j=0}^{N} e_{\lambda+j} \vee \bigvee_{j=0}^{N} f_{\lambda+j} \leq \bigvee_{j=0}^{2N} g_{\lambda+j} \vee \bigvee_{j=0}^{2N} h_{\lambda+j} \quad \text{and hence}$$

$\bigvee_{\alpha\in A} e_\alpha \vee \bigvee_{\alpha\in A} f_\alpha \leq \bigvee_\lambda (\bigvee G_\lambda \vee \bigvee H_\lambda) \leq \bigvee_{\alpha\in A} e_\alpha \vee \bigvee_{\alpha\in A} f_\alpha$, where the join over λ is taken over all limit ordinals in A, and the second inequality follows from the way in which the $g_{\lambda+n}$ and $h_{\lambda+n}$ were constructed. By lemma 2.4, the $g_{\lambda+n}$ and $h_{\lambda+n}$ are simple elements. Now, $|g_{\lambda+n}| = |e_{\lambda+n}| \vee |f_{\lambda+2n} \wedge |e_{\lambda+n}|'| = |e_{\lambda+n}| \vee (|f_{\lambda+2n}| \wedge |e_{\lambda+n}|')$ $= u \wedge (|e_{\lambda+n}| \vee |e_{\lambda+n}|') = u$, and

$|h_{\lambda+n}| = |f_{\lambda+n} \wedge |e_{\lambda+q(n)}|| \vee |f_{\lambda+2n+1} \wedge |e_{\lambda+q(n)}|'|$

$= (|f_{\lambda+n}| \wedge |e_{\lambda+q(n)}|) \vee (|f_{\lambda+2n+1}| \wedge |e_{\lambda+q(n)}|') = (u \wedge |e_{\lambda+q(n)}|) \vee$ $(u \wedge |e_{\lambda+q(n)}|') = u \wedge (|e_{\lambda+q(n)}| \vee |e_{\lambda+q(n)}|') = u$. The cardinality of $G = \bigcup_\lambda G_\lambda$ and $H = \bigcup_\lambda H_\lambda$ is c. Since c is an infinite cardinal, the cardinality of $G \cup H$ is also c. All that remains is to show that $G \cup H$ is an orthogonal family. This can be reduced to the following three cases. Case I, to show that $m \neq n$ implies $g_{\lambda+m} \perp g_{\lambda+n}$. This follows from $e_{\lambda+m} \perp e_{\lambda+n}$, $e_{\lambda+n} \perp f_{\lambda+2m}$, $f_{\lambda+2n} \perp e_{\lambda+m}$, and $f_{\lambda+2n} \perp f_{\lambda+2m}$. Case II, to show that $m \neq n$ implies $h_{\lambda+m} \perp h_{\lambda+n}$. If $m \neq 2n+1$ and $n \neq 2m+1$, this case follows from $f_{\lambda+n} \perp f_{\lambda+m}$, $f_{\lambda+n} \perp f_{\lambda+2m+1}$, $f_{\lambda+2n+1} \perp f_{\lambda+m}$, and $f_{\lambda+2n+1} \perp f_{\lambda+2m+1}$. The situations where $m = 2n+1$ or $n = 2n+1$ are symmetric, so we may handle both at once. Suppose that $m = 2n+1$, then $n \neq 2m+1$, and $|e_{\lambda+q(n)}| = |e_{\lambda+q(2n+1)}|$. Hence the result follows from $f_{\lambda+n} \perp f_{\lambda+m}$, $f_{\lambda+n} \perp f_{\lambda+2m+1}$, $f_{\lambda+2m+1} \perp f_{\lambda+2n+1}$, and $|e_{\lambda+q(n)}|' \perp |e_{\lambda+q(2n+1)}|$. Case III, to show that $g_{\lambda+n} \perp h_{\lambda+m}$. Now $2n \neq 2m+1$, for one is even and the other is odd. Hence, if $m \neq 2n$, the result follows from $e_{\lambda+n} \perp f_{\lambda+m}$, $e_{\lambda+n} \perp f_{\lambda+2m+1}$, $f_{\lambda+2n} \perp f_{\lambda+2m+1}$, and $f_{\lambda+2n} \perp f_{\lambda+m}$.

If $m = 2n$, then, since $q(2n) = n$, the result follows from $e_{\lambda+n} \perp f_{\lambda+m}$, $e_{\lambda+n} \perp f_{\lambda+2m+1}$, $f_{\lambda+2n} \perp f_{\lambda+2m+1}$, and $|e_{\lambda+n}|' \perp |e_{\lambda+q(2n)}|$.

Lemma 2.24. Suppose that $(L, |\ |)$ is piecewise semimodular, and that a is an element with uniform multiplicity c, then a is the union of c, orthogonal, simple elements $\{a_\mu\}$, such that $|a| = |a_\mu|$ for all μ.

Proof: If a is finite and c is a finite cardinal, we have the result in lemma 2.19, thus, we can suppose that a is purely infinite and c is an infinite cardinal. Let $\{e_\alpha\}$ be a family of c, orthogonal, simple elements such that $\bigvee_\alpha e_\alpha = a$. Suppose that the set has been well ordered, where α is in the indexing set A. Let, as in lemma 2.22, f_1 be a simple subelement of a such that $e_1 \trianglelefteq f_1$, $|f_1| = |\bigvee_\alpha e_\alpha| = |a|$, and $f_1 C e_\alpha$ for all α. Consider the family $\{e_\alpha \wedge f_1'\}$. By lemma 2.22 again, let f_2 be a simple subelement of $\bigvee_\alpha (e_\alpha \wedge f_1') \trianglelefteq a$, such that $|f_2| = |\bigvee_\alpha (e_\alpha \wedge f_1')|$, f_2 contains the first nonzero element of the set $\{e_2 \wedge f_1', e_3 \wedge f_1', \ldots, e_\alpha \wedge f_1', \ldots\}$, and f_2 is the union of mutually unrelated subelements of the family $\{e_\alpha \wedge f_1'\}$. It follows that $f_1 \perp f_2$, $e_1 \vee e_2 \trianglelefteq f_1 \vee f_2$, and f_2 is the join of mutually unrelated subelements of the family $\{e_\alpha\}$, that is, $f_2 C e_\alpha$ for all α. Suppose that f_α has been chosen for all $\alpha \prec \delta$ (δ is a fixed ordinal) such that $\{f_1, f_2, \ldots, f_\alpha, \ldots : \alpha \prec \delta\}$ is an orthogonal family of simple subelements of a with $\bigvee_{\gamma \trianglelefteq \alpha} e_\gamma \trianglelefteq \bigvee_{\gamma \trianglelefteq \alpha} f_\gamma$ for all $\alpha \prec \delta$, $|f_\alpha| = |\bigvee_\gamma [e_\gamma \wedge (\bigvee_{\beta \prec \alpha} f_\beta)']|$ for all $\alpha \prec \delta$, and $f_\alpha C e$ for all $\alpha \prec \delta$ and all β in A. Let $h_\delta = \bigvee_{\alpha \prec \delta} f_\alpha$, and consider the family $\{e_\alpha \wedge h_\delta'\}$. If $e_\alpha \wedge h_\delta' = 0$ for all $\alpha \in A$, then $0 = \bigvee_\alpha (e_\alpha \wedge h_\delta') = h_\delta' \wedge \bigvee_\alpha e_\alpha = h_\delta' \wedge a$.

Since $h_\delta \leq a$, we must have $h_\delta = a$ and the induction may stop. Otherwise, let E_δ be the first nonzero element of the family $\{e_\alpha \wedge h'_\delta\} = \{e_\delta \wedge h'_\delta,\ e_{\delta+1} \wedge h'_\delta,\ \ldots\}$. Now let f_δ be a simple subelement of $V_\alpha(e_\alpha \wedge h'_\delta) \leq a$ which contains E_δ and has the other properties specified in lemma 2.22. It follows that f_δ is the union of mutually unrelated subelements of the family $\{e_\alpha\}$ and, hence, $f_\delta C e_\alpha$ for all $\alpha \in A$. Let $\{f_\alpha\}$ be the family of simple subelements of a obtained when the induction is completed and $\{e_\alpha\}$ is finally exhausted. Since $f_\alpha \leq h'_\alpha$, it follows that the f_α are mutually orthogonal. Furthermore, $a = V_\alpha e_\alpha \leq V_\alpha f_\alpha \leq a$, that is $V_\alpha f_\alpha = a$. By the inductive process it is clear that the cardinality of $\{f_\alpha\}$ is less than or equal to c. By the uniform multiplicity of a, the cardinality of $\{f_\alpha\}$ is greater than or equal to c. Thus, the cardinality of $\{f_\alpha\}$ is c. Now if $\alpha \leq \beta$, $|f_\alpha| \geq |f_\beta|$ and $|f_1| = |a|$. If $|f_\alpha| = |a|$ for all α, we are done. Otherwise, let λ be the first ordinal such that $|f_\lambda| < |a|$, then $a \wedge |f_\lambda|' \neq 0$ and $|f_\alpha| = |a|$ for all $\alpha < \lambda$. $\{f_\alpha \wedge |f_\lambda|' : \alpha < \lambda\}$ is an orthogonal set of simple elements. Also, $a \wedge |f_\lambda|' = |f_\lambda|' \wedge V_\alpha f_\alpha = V_\alpha(|f_\lambda|' \wedge f_\alpha) = V_{\alpha<\lambda}(|f_\lambda|' \wedge f_\alpha)$. By the uniform multiplicity of a, the family $\{f_\alpha : |f_\alpha| = |a|\} = \{f_\alpha : \alpha < \lambda\}$ must have cardinality c. On the other hand, $\{f_\alpha : \lambda \leq \alpha\}$ will have cardinality less than or equal to c. The result now follows from lemma 2.23.

Definition 2.25. Let $(L, |\ |)$ be piecewise semimodular, and let a be a purely infinite element thereof. For every invariant e, such that $e \leq |a|$, let c(e) be the smallest cardinal number such that

$a \wedge e$ can be expressed as the union of $c(e)$, orthogonal, simple elements. Let $\{c_\gamma\}$ be the well-ordered family of cardinal numbers which thus appear. If e_1 is the union of all invariant elements e such that $c(e) = c_1$, then $c(e_1) = c_1$. To prove this, let $\{e_\alpha\}$ be a maximal, orthogonal subset of the set of all invariant elements e such that $c(e) = c_1$. By the maximality of the collection, $e_1 = \bigvee_\alpha e_\alpha$. For each α, $a \wedge e_\alpha = \bigvee_\beta a_{\alpha\beta}$ where each set $\{a_{\alpha\beta} : \beta \in B\}$ is an orthogonal collection of simple element with cardinality c_1, such that $|a_{\alpha\beta}| \leq e_\alpha$. Since each set has the same cardinality, we may index them by the same set B. By lemma 2.4, $a_\beta = \bigvee_\alpha a_{\alpha\beta}$ is simple for each $\beta \in B$. Moreover, $\{a_\beta : \beta \in B\}$ is an orthogonal set of cardinality c_1. Finally, $\bigvee_\beta a_\beta = \bigvee_\beta \bigvee_\alpha a_{\alpha\beta} = \bigvee_\alpha \bigvee_\beta a_{\alpha\beta} = \bigvee_\alpha (a \wedge e_\alpha) = a \wedge \bigvee_\alpha e_\alpha = a \wedge e = a$. Hence, $c(e_1) \leq c_1$, but $c_1 \leq c(e)$ for all e. Thus, $c(e_1) = c_1$. Now it can be seen that $a \wedge e_1$ has uniform multiplicity c_1, and hence by lemma 2.24, $a \wedge e_1$ is the union of c_1, orthogonal, simple elements, each of which has e_1 as its hull, that is, the orthogonal, simple elements are all equivalent. We next work on $a \wedge e_1'$, where the smallest cardinal number appearing is c_2, and where we repeat the above construction. We thus define inductively a well-ordered set of invariant elements $\{e_\gamma\}$ and cardinal numbers $\{c_\gamma\}$ such that on e_γ, a is the union of c_γ, orthogonal, simple elements, and also such that there is no way of expressing $a \wedge e_\gamma$ as the union of fewer than c_γ, orthogonal, simple elements. The family of pairs $\{(e_\gamma, c_\gamma)\}$ is the multiplicity family for a. Now, let a be an arbitrary element of $(L, |\ |)$, and let $a = a_1 \vee a_2$ be the unique decomposition of lemma 2.21.

Let $\{(e_\gamma, c_\gamma)\}$ be the multiplicity family of a_2, and $\{u_n\}$ be the multiplicity sequence of a_1. Define the collection

$A = \{(u_n, n)\colon u_n \neq 0$ and u_n an element of the multiplicity sequence of $a_1\}$.

Re-index the set $A \cup \{(e_\gamma, c_\gamma)\}$ by an ordered indexing set I such that, if $\lambda, \gamma \in I$ with $\gamma \leq \lambda$, then $c_\gamma \leq c_\lambda$. The resulting collection is called the multiplicity family of a, and is denoted by MF(a).

CHAPTER III

M-SYMMETRY AND PIECEWISE SEMIMODULARITY

Definition 3.1. If ϕ_1 and ϕ_2 are quantifiers on L, we write $\phi_1 \trianglelefteq \phi_2$ whenever $\phi_1\phi_2 = \phi_1$, that is $(a\phi_1)\phi_2 = a\phi_1$ for all $a \in L$.

Lemma 3.2. Let ϕ_1 and ϕ_2 be two quantifiers on L with $\phi_1 \trianglelefteq \phi_2$, then

(i) ϕ_1-invariant elements are ϕ_2-invariant,

(ii) ϕ_1-simple elements are ϕ_2-simple,

(iii) if (L,ϕ_1) is type I, then so is (L,ϕ_2).

Proof: (i) is clear and (iii) follows immediately from (ii). As for the proof of (ii), let a be ϕ-simple and $b \trianglelefteq a$. Now, $b = a \wedge b\phi_1$. By (i), $b\phi_2 \trianglelefteq b\phi_1$. Thus, $b \trianglelefteq b\phi_2$ and $b \trianglelefteq a$ implies that $b \trianglelefteq a \wedge b\phi_2 \trianglelefteq a \wedge b\phi_1 = b$. Hence, $b = a \wedge b\phi_2$ and a is ϕ_2-simple.

Lemma 3.3. (i) If ϕ_s is the simple quantifier, then (L,ϕ_s) is type I if and only if L is atomic.

(ii) If L is atomic and ϕ is any quantifier on L, then (L,ϕ) is type I.

Proof: For (i) suppose that (L,ϕ_s) is type I, and that a is a ϕ_s-simple element. If $b \trianglelefteq a$, then $b = a \wedge b\phi_s$. Now, $b\phi_s$ is 0 or 1 and, hence, $b = a$ or $b = 0$. Hence, all simple elements of (L,ϕ_s) are atomic. Thus, every element of L is the orthogonal supremum of atoms, that is, L is atomic. On the other hand, suppose that L is atomic.

Now every atom of L is ϕ_s-simple. Thus, every element of L is the orthogonal supremum of ϕ_s-simple elements. For (ii), let ϕ be any quantifier on L, then $0\phi = 0$ and $1\phi = 1$. Hence, $\phi_s \leq \phi$. The result now follows from lemma 3.2 part (iii) and lemma 3.3 part (i).

Lemma 3.4. Let L be atomic and let ϕ_1 and ϕ_2 be two center valued quantifiers on L. Then (L,ϕ_1) is piecewise semimodular if and only if (L,ϕ_2) is piecewise semimodular.

Proof: Since the relations $\phi_s \leq \phi_1$ and $\phi_s \leq \phi_2$ hold, it is enough to prove the lemma for comparable quantifiers. Thus, suppose that $\phi_1 \leq \phi_2$. Since L is an atomic lattice, the range of ϕ_1 and the range of ϕ_2 are atomic lattices. First, suppose that (L,ϕ_2) is piecewise semimodular. Let a be an element of L which has uniform multiplicity with respect to ϕ_1. Without loss of generality, we may suppose that $a\phi_1$ is an atom in the range of ϕ_1. Suppose that $a = \bigvee_\alpha a_\alpha = \bigvee_\beta b_\beta$, where $\{a_\alpha\}$ and $\{b_\beta\}$ are two families of nonzero, orthogonal, ϕ_1-simple elements with $a_\alpha\phi_1 = b_\beta\phi_1 = a\phi_1$ for all α and β. Since the only ϕ_1-invariant sub-elements of $a\phi_1$ are 0 and $a\phi_1$, it follows that the ϕ_1-simple elements a_α and b_β are atoms in L. Let $\{f_\gamma\}$ be an orthogonal family of ϕ_2-invariant elements such that $\bigvee_\gamma f_\gamma = a\phi_1$ (note that by lemma 3.2 part (i) $a\phi_1$ is ϕ_2-invariant) and such that each f_γ is an atom in the range of ϕ_2. For each γ, define the sets $A_\gamma = \{a_\alpha: a_\alpha \wedge f_\gamma \neq 0\}$ and $B_\gamma = \{b_\beta: b_\beta \wedge f_\gamma \neq 0\}$. Since a_α is an atom in L, $a_\alpha \wedge f_\gamma$ is either 0 or a_α. If $a_\alpha \wedge f_\gamma \neq 0$, then $(a_\alpha \wedge f_\gamma)\phi_2 = a_\alpha\phi_2 \wedge f_\gamma \neq 0$. Since f_γ is an atom in the range of ϕ_2, $a_\alpha \wedge f_\gamma \neq 0$ implies that $(a_\alpha \wedge f_\gamma)\phi_2 = f_\gamma$. Thus, for $a_\alpha \wedge f_\gamma \neq 0$, we have $a_\alpha \wedge f_\gamma = a_\alpha \leq a_\alpha\phi_2 = f_\gamma$. Since the f_γ

are orthogonal, each a_α is a member of, at most, one of the sets A_γ. Now, $a_\alpha = a_\alpha \wedge \bigvee_\gamma f_\gamma = \bigvee_\gamma (a_\alpha \wedge f_\gamma)$. Hence, $a_\alpha \wedge f_\gamma \neq 0$ for at least one γ. Thus, we have shown that each a_α is a member of one and only one of the sets A_γ. Similarly, each b_β is a member of one and only one of the sets B_γ. By lemma 3.2 part (ii), all of the a_α and b_β are ϕ_2-simple elements of L. By the piecewise semimodularity of L, the cardinality of A_γ is equal to the cardinality of B_γ for each γ. Since $\{a_\alpha\}$ is the disjoint union of the sets A_γ and $\{b_\beta\}$ is the disjoint union of the sets B_γ, it follows that the cardinality of $\{a_\alpha\}$ is equal to the cardinality of $\{b_\beta\}$. This proves the implication of the lemma in one direction. For the second direction we suppose that (L,ϕ_1) is piecewise semimodular. Now let a be a nonzero element of uniform multiplicity with respect to ϕ_2. Without loss of generality, we may suppose that $a\phi_2$ is an atom in the range of ϕ_2. Now suppose that $a = \bigvee_\alpha a_\alpha$ where $\{a_\alpha\}$ is an orthogonal collection of c, nonzero, ϕ_2-simple elements with $a_\alpha\phi_2 = a\phi_2$ for all α. Since $a\phi_2$ is an atom in the range of ϕ_2, each a_α is an atom in L. Thus, each a_α is ϕ_1-simple. Now, for each α, $a_\alpha\phi_1 = \bigwedge\{e: a_\alpha \leq e \text{ and } e\phi_1 = e\} \geq \bigwedge\{f: a_\alpha \leq f \text{ and } f\phi_2 = f\} = a_\alpha\phi_2 = a\phi_2$. Hence, if $a_\alpha \leq e$ and $e\phi_1 = e$, then $a \leq e$. Thus, $a_\alpha\phi_1 = \bigwedge\{e: a_\alpha \leq e \text{ and } e\phi_1 = e\} = \bigwedge\{e: a \leq e \text{ and } e\phi_1 = e\} = a\phi_1$. We then have $a = \bigvee_\alpha a_\alpha$, where $\{a_\alpha\}$ is a collection of c, nonzero, orthogonal, ϕ_1-simple elements with $a_\alpha\phi_1 = a\phi_1$. Since (L,ϕ_1) is piecewise semimodular, it follows that the cardinal number c is unique.

Theorem 3.5. If L is a complete, atomic, M-symmetric, orthomodular lattice, and if ϕ is a center valued quantifier on L, then (L,ϕ) is type I and piecewise semimodular.

Proof: That (L,ϕ) is type I, follows from lemma 3.3. By theorem B of M. Donald Maclaren's paper [11], there is an equivalence relation $\sim$ on L, such that $(L,\sim)$ is a dimension lattice. Thus, according to L. H. Loomis [9], there is a center valued quantifier ϕ_0, associated with $(L,\sim)$, such that (L,ϕ_0) is piecewise semimodular. Hence, by lemma 3.4, (L,ϕ) is piecewise semimodular.

Remarks and Definition 3.6. By lemma 3.4, piecewise semimodularity is independent of which center valued quantifier on a given lattice one looks at. Thus, let L be a complete, atomic, orthomodular lattice. We say that L is piecewise semimodular if there is some center valued quantifier, ϕ, on L such that (L,ϕ) is piecewise semimodular. In this manner we may restate theorem 3.5 in the following way.

Theorem 3.7. If L is a complete, atomic, M-symmetric, orthomodular lattice, then L is piecewise semimodular.

Definition 3.8. If L_1 and L_2 are two complete, orthomodular lattices, we define the direct product of L_1 and L_2 to be $L_1 \times L_2 = \{(a,b)\colon a \in L_1,\ b \in L_2\}$ where the operations of supremum, infimum, and orthocomplementation are done coordinatewise.

That $L_1 \times L_2$ is a complete, orthomodular lattice and other elementary properties of direct products may be found in Orthomodular Lattice Theory by David Foulis [2].

Definition 3.9. If ϕ_1 and ϕ_2 are center valued quantifiers on L_1 and L_2 respectively, we define the direct product of ϕ_1 and ϕ_2 to be the quantifier on $L_1 \times L_2$ whose range is $\{(e,f)\colon e = e\phi_1,\ f = f\phi_2\}$. The direct product of ϕ_1 and ϕ_2 is indicated by $\phi_1 \times \phi_2$.

That $\phi_1 \times \phi_2$ is a center valued quantifier, follows from lemma 1.2 and elementary properties of lattice direct products.

Lemma 3.10. If $L_1 \times L_2$ is the direct product of two, complete, orthomodular lattices L_1 and L_2, and if ϕ is a center valued quantifier on $L_1 \times L_2$, then there exists unique center valued quantifiers ϕ_1 and ϕ_2 on L_1 and L_2 respectively, such that $\phi = \phi_1 \times \phi_2$ and $(a,b)\phi = (a\phi_1, b\phi_2)$.

Proof: Take ϕ_1 to be the quantifier whose range is $\{e: (e,0) = (e,0)\phi\}$ and take ϕ_2 to be the quantifier whose range is $\{f: (0,f) = (0,f)\phi\}$. The remaining details are easily checked.

Lemma 3.11. If $\phi = \phi_1 \times \phi_2$ is a center valued quantifier on $L = L_1 \times L_2$, where L_1 and L_2 are complete, orthomodular lattices, then (L,ϕ) is piecewise semimodular if and only if (L,ϕ_1) and (L,ϕ_2) are both piecewise semimodular.

Proof: Take $a \in L_1$ and $\{a_\alpha\}$ be a family of orthogonal, nonzero, ϕ_1-simple elements of L_1 such that $a = \bigvee_\alpha a_\alpha$ and $a\phi_1 = a_\alpha\phi_1$ for all α. Now, $(a,0) = \bigvee_\alpha (a_\alpha,0)$, $(a,0)\phi = (a\phi_1,0) = (a_\alpha\phi_1,0) = (a_\alpha,0)\phi$, and each $(a_\alpha,0)$ is ϕ-simple. Thus, if (L,ϕ) is piecewise semimodular, the cardinality of the family $\{(a_\alpha,0)\}$ is some fixed cardinal number and, hence, the cardinality of the family $\{a_\alpha\}$ is also the same unique cardinal number. Therefore, (L_1,ϕ_1) is piecewise semimodular. Similarly, if (L,ϕ) is piecewise semimodular, then so is (L_2,ϕ_2). Now, suppose that (L_1,ϕ_1) and (L_2,ϕ_2) are piecewise semimodular. Let $(a,b) \in L$ have uniform multiplicity. Let $\{(a_\alpha,b_\alpha)\}$ be a collection of nonzero, orthogonal, ϕ-simple elements of L with $(a,b)\phi = (a_\alpha,b_\alpha)\phi$, and $(a,b) = \bigvee_\alpha (a_\alpha,b_\alpha)$. Now, $(a\phi_1,b\phi_2) = (a,b)\phi = (a_\alpha,b_\alpha)\phi = (a_\alpha\phi_1,b_\alpha\phi_2)$ for all α, so that, $a\phi_1 = a_\alpha\phi_1$ and $a\phi_2 = b_\alpha\phi_2$ for all α. Since $(a,b) \neq 0$, at least one of $a \neq 0$, $b \neq 0$ holds. We suppose that $a \neq 0$, the other

case is similar. Since $a \neq 0$, all of the a_α have nonzero hulls and, hence, $a_\alpha \neq 0$ for all α. Now, $a = \bigvee_\alpha a_\alpha$ and $\{a_\alpha\}$ is an orthogonal collection of nonzero, ϕ-simple elements of L_1. By the piecewise semimodularity of L_1, the cardinality of the family $\{a_\alpha\}$ is unique. Hence, the cardinality of the family $\{(a_\alpha, b_{a'})\}$ is also unique, that is, (L,ϕ) is piecewise semimodular.

Corollary 3.12: If L_1 and L_2 are complete, atomic, orthomodular lattices, then $L_1 \times L_2$ is piecewise semimodular if and only if L_1 and L_2 are both piecewise semimodular.

Definition 3.13. Let L_1 and L_2 be two distinct orthomodular lattices. If we identify the two zero elements and the two unit elements, and let $L = L_1 \cup L_2$, we can define a partial order on L by the rule that $a \leq b$ in L if and only if $a \leq b$ in L_1 or L_2. We denote L by $L_1 \boxplus L_2$ and L is called the disjoint sum or horizontal sum of the summands L_1 and L_2. If a and b are elements of the same summand, then the supremum and infimum of a and b have the same value they have in the original summand. If and and b are in different summands, then $a \vee b = 1$ and $a \wedge b = 0$ in L. The orthocomplement of an element retains its original value. Under these operations, L is an orthomodular lattice. If both L_1 and L_2 contain more than two elements, then $L = L_1 \boxplus L_2$ is called a nontrivial disjoint sum.

Lemma 3.14. Let $L = L_1 \boxplus L_2$ be a nontrivial disjoint sum, and let ϕ be a center valued quantifier on L. The following are equivalent:

(i) (L,ϕ) is type I,

(ii) L is atomic,

(iii) L_1 and L_2 are both atomic.

Proof: That (i) and (ii) are equivalent follows from the fact that a nontrivial disjoint sum is irreducible, and lemma 3.3. That (ii) and (iii) are equivalent follows from the definition of disjoint sum.

Definition 3.15. Let ϕ_s denote the simple quantifier on a complete, atomic, orthomodular lattice L. Let (L,ϕ_s) be piecewise semimodular. Since the only invariant elements are 0 and 1, every element of (L,ϕ_s) has uniform multiplicity. By piecewise semimodularity, every element of (L,ϕ_s) is the orthogonal union of a unique number of atoms. We define a function d from L into the set of cardinal numbers such that $d(a) = c$, if c is the unique number of orthogonal atoms associated with a by piecewise semimodularity.

Lemma 3.16. Let $L = L_1 \boxplus L_2$ and let ℓ_1 and ℓ_2 be the unit elements of L_1 and L_2 respectively. L is piecewise semimodular if and only if L_1 and L_2 are both piecewise semimodular and $d(\ell_1) = d(\ell_2)$.
Proof: Suppose that L_1 and L_2 are piecewise semimodular, and that $d(\ell_1) = d(\ell_2)$. Since orthocomplements remain in the same summand, all of the calculations for nonunits are clear. Also, any orthogonal collection of atoms whose join is ℓ has cardinality $d(\ell_1)$ or $d(\ell_2)$. Since $d(\ell_1) = d(\ell_2)$, it follows that (L,ϕ_s) is piecewise semimodular, then it is clear that (L_1,ϕ_{1s}) and (L_2,ϕ_{2s}) are each piecewise semimodular, where ϕ_{1s} and ϕ_{2s} denote the simple quantifiers on L_1 and L_2 respectively, and it is clear that $d(\ell_1) = d(\ell_2)$. The result now follows from lemma 3.4.

CHAPTER IV

DIMENSION LATTICES

In this chapter $(L, |\ |)$ will always denote a type I, piecewise semimodular, monadic lattice with a center valued quantifier.

Definition 4.1. If a and b are elements of $(L, |\ |)$ such that $MF(a) = MF(b)$, then we say that a and b are equivalent and write $a \sim b$.

Thus, with $(L, |\ |)$ is associated a pair $(L, \sim)$. In this chapter we will show that the specified properties of $(L, |\ |)$ imply that $(L, \sim)$ is a dimension lattice.

Lemma 4.2. If $a \sim 0$, then $a = 0$. That is, condition A is satisfied in $(L, \sim)$.

Proof: 0 is finite and its multiplicity sequence is $\{0, 0, 0, \ldots\}$. Thus, $MF(0) = \emptyset$ the empty set. Now, if $MF(a) = \emptyset$, we have $|a| = \bigvee \emptyset = 0$. Hence, $a = 0$.

Lemma 4.3. If a and b are not orthogonal, then there exist nonzero elements $a_1 \leq a$ and $b_1 \leq b$ such that $a_1 \sim b_1$. That is, condition D is satisfied in $(L, \sim)$.

Proof: If $|a| \wedge b = 0$, we would have $0 = ||a| \wedge b| = |a| \wedge |b|$, which implies that $|a| \perp |b|$ and, thus, $a \perp b$. Since a and b are not orthogonal, we have $|a| \wedge b \neq 0$ and $|a| \wedge |b| \neq 0$. By lemma 2.5, there are simple elements a_2, b_1 such that $a_2 \leq a$, $|a_2| = |a|$, $b_1 \leq b \wedge |a|$, and $|b_1| = |b \wedge |a|| = |b| \wedge |a|$. Let $a_1 = a_2 \wedge |a| \wedge |b|$

$= a_2 \wedge |b|$, then $|a_1| = |a_2| \wedge |b| = |a| \wedge |b|$. Since the hulls of a_1 and b_1 are nonzero, a_1 and b_1 are nonzero. Furthermore, $MF(a) = \{(|a| \wedge |b|, 1)\} = MF(b)$. Therefore, $a_1 \sim b_1$.

Lemma 4.4. If e is invariant and $a \sim b$, then $a \wedge e \sim b \wedge e$.

Proof: Let $\{(u_\lambda, c_\lambda)\} = MF(a) = MF(b)$. Suppose that $u_\lambda \wedge e \neq 0$. Now, $a \wedge u_\lambda = \bigvee_\alpha a_\alpha$ where $\{a_\alpha\}$ is an orthogonal collection of c_λ, nonzero, simple elements with $|a_\alpha| = u_\lambda$. $a \wedge u_\lambda \wedge e = e \wedge \bigvee_\alpha a_\alpha = \bigvee_\alpha (a_\alpha \wedge e)$. $|a_\alpha \wedge e| = |a_\alpha| \wedge e = u_\lambda \wedge e \neq 0$, therefore, $a_\alpha \wedge e \neq 0$ for all α. Hence, $a \wedge u_\lambda \wedge e$ is the join of c_λ, orthogonal, nonzero, simple elements with hull $u_\lambda \wedge e$. It follows that

$MF(a \wedge e) = \{(u_\lambda \wedge e, c_\lambda): (u_\lambda, c_\lambda) \in MF(a), \text{ and } u_\lambda \wedge e \neq 0\}$.

Similarly, we can show that this same set of ordered pairs is the multiplicity family of $b \wedge e$.

Lemma 4.5. Let L be atomic. If $a \wedge e \sim b \wedge e$ for all atoms e, in the range of $||$, then $a \sim b$.

Proof: Suppose that $(u,c) \in MF(a)$ and that $\{e_\gamma\}$ is an orthogonal collection of atoms in the range of $||$ such that $\bigvee_\gamma e_\gamma = u$. Now, $a \wedge u$ is the join of c, nonzero, orthogonal, simple elements $\{a_\alpha\}$ with $|a_\alpha| = u$. Thus, $\bigvee_\alpha a_\alpha = a \wedge u = a \wedge \bigvee_\gamma e_\gamma = \bigvee_\gamma (a \wedge e_\gamma)$, and $e_\lambda \wedge a = e_\lambda \wedge \bigvee_\gamma (a \wedge e_\gamma) = e_\lambda \wedge \bigvee_\alpha a_\alpha = \bigvee_\alpha (a_\lambda \wedge e_\gamma)$ for any $e_\lambda \in \{e_\gamma\}$. Now, $|a_\alpha \wedge e_\lambda| = |a_\alpha| \wedge e_\lambda = u \wedge e_\lambda$. Thus, $e_\lambda \wedge a$ is the join of c, nonzero, orthogonal, simple elements with hull e_λ. Because this holds for all $e_\lambda \in \{e_\gamma\}$, it follows from lemma 2.4 that $a \wedge u$ is the join of c, nonzero, orthogonal, simple elements with hull u. Hence, there is an invariant element v, such that $(v,c) \in MF(b)$ and $u \leq v$. If $u \prec v$, there

is an atom e, in the range of $||$, such that $a \wedge e$ is not equivalent to $b \wedge e$. This contradiction implies that $u = v$, that is, $(u,c) \in MF(b)$. By a similar argument, if $(v,c) \in MF(b)$, then $(v,c) \in MF(a)$. Hence, $MF(a) = MF(b)$.

Lemma 4.6. Let L be atomic, and let $\{a_\alpha: \alpha \in I$ and $\{b_\alpha: \alpha \in I\}$ be two families of mutually unrelated elements over the same indexing set I. If $a_\alpha \sim b_\alpha$ for all $\alpha \in I$, then $\bigvee_\alpha a_\alpha \sim \bigvee_\alpha b_\alpha$.

Proof: Let e be an atom in the range of $|\ |$. If $e \leq (\bigvee_\alpha |a_\alpha|)' = (\bigvee_\alpha |b_\alpha|)'$, then $e \wedge \bigvee_\alpha a_\alpha = e \wedge \bigvee_\alpha b_\alpha = 0$. If $e \leq \bigvee_\alpha |a_\alpha| = \bigvee_\alpha |b_\alpha|$, then there is some α_o such that $e \leq |a_{\alpha_o}| = |b_{\alpha_o}|$. Now, $e \bigvee_\alpha a_\alpha = \bigvee_\alpha (a_\alpha \wedge e) = a_{\alpha_o} \wedge e$, and $e \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha (b_\alpha \wedge e) = b_{\alpha_o} \wedge e$. By lemma 4.4, $a_{\alpha_o} \sim b_{\alpha_o}$, implies that $a_{\alpha_o} \wedge e \sim b_{\alpha_o} \wedge e$. Hence, $e \wedge \bigvee_\alpha a_\alpha \sim e \wedge \bigvee_\alpha b_\alpha$ for all atoms e, in the range of $|\ |$. By lemma 4.5, $\bigvee_\alpha a_\alpha \sim \bigvee_\alpha b_\alpha$.

Lemma 4.7. Let L be atomic. If $\{a_\alpha: \alpha \in E\}$ and $\{b_\alpha: \alpha \in I\}$ are orthogonal families over the same indexing set I, and if $a_\alpha \sim b_\alpha$ for all $\alpha \in I$, then $\bigvee_\alpha a_\alpha \sim \bigvee_\alpha b_\alpha$. That is, $(L,\sim)$ satisfies condition C.

Proof: Let e be an atom in the range of $|\ |$. Then $a_\alpha \wedge e \sim b_\alpha \wedge e$ by lemma 4.4, and $|b_\alpha \wedge e| = |b_\alpha| \wedge e = |a_\alpha| \wedge e = |a_\alpha \wedge e| = 0$ or e. Hence, $MF(a_\alpha \wedge e) = MF(b_\alpha \wedge e)$ is either the empty set or the singleton set $\{(e,c_\alpha)\}$, for some cardinal number c_α. Now, $a_\alpha \wedge e$ is the join of c_α, nonzero, orthogonal, simple subelements of a_α, $\{a_{\alpha\gamma}: \gamma \in I_\alpha\}$. Hence, $\bigvee_\alpha (a_\alpha \wedge e) = e \wedge \bigvee_\alpha a_\alpha = \bigvee_\alpha (\bigvee_\gamma a_{\alpha\gamma})$. Since $\{a_\alpha\}$ is an orthogonal family, $\bigcup_\alpha \{a_{A\gamma}: \gamma \in I_\alpha\}$ is also an orthogonal family. The elements of this set are nonzero, simple elements with $|a_{\alpha\gamma}| = e$ for all α and γ.

Furthermore, the cardinality of the set is $\Sigma_\alpha c_\alpha$. Similarly, we may show that $e \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha (e \wedge b_\alpha)$ is the join of $\Sigma_\alpha c_\alpha$, nonzero, orthogonal, simple elements with hull e. Therefore, $MF(e \wedge \bigvee_\alpha a_\alpha) = MF(e \wedge \bigvee_\alpha b_\alpha)$. This holds for all atoms e, in the range of $|\ |$, so that by lemma 4.5, $\bigvee_\alpha a_\alpha \sim \bigvee_\alpha b_\alpha$.

Lemma 4.8. Let L be atomic. If $\{a_\alpha: \alpha \in I\}$ is an orthogonal family and if $b \sim a = \bigvee_\alpha a_\alpha$, then there is an orthogonal family $\{b_\alpha: \alpha \in I\}$, such that $b = \bigvee_\alpha b_\alpha$ and $a_\alpha \sim b_\alpha$ for all α. That is, $(L,\sim)$ satisfies condition B'.

Proof: Let $\{e_\gamma\}$ be an orthogonal family of atoms in the range of $|\ |$, such that $\bigvee_\gamma e_\gamma = 1$. By lemma 4.4, for each λ, $b \wedge e_\lambda \sim a \wedge e_\lambda = e_\lambda \wedge \bigvee_{\alpha \in I} a_\alpha = \bigvee_{\alpha \in I} (e_\lambda \wedge a_\alpha)$. Since e_λ is an atom in the range of $|\ |$, the multiplicity family of $e_\lambda \wedge a_\alpha$ is either empty or the singleton set $\{(e_\lambda, c_{\alpha\lambda})\}$ for an appropriate cardinal number $c_{\alpha\lambda}$. Thus, there is a family $\{a_{\alpha\lambda\gamma}: \gamma \in I_{\alpha\lambda}\}$ of $c_{\alpha\lambda}$ nonzero, orthogonal, simple elements with hull e_λ such that $a_\alpha \wedge e_\lambda = \bigvee_\gamma a_{\alpha\lambda\gamma}$ (take $c_{\alpha\lambda} = 0$ if $MF(a_\alpha \wedge e_\lambda)$ is empty). Now, $a \wedge e_\lambda = \bigvee_\alpha \bigvee_\gamma a_{\alpha\lambda\gamma}$ is the join of $\Sigma_\alpha c_{\alpha\lambda}$, nonzero, orthogonal, simple elements with hull e_λ, likewise for $b \wedge e_\lambda$. Divide the nonzero, orthogonal, simple elements with hull e_λ, whose join is $b \wedge e_\lambda$, into sets $B_{\alpha\lambda} = \{b_{\alpha\lambda\gamma}: \gamma \in I_{\alpha\lambda}\}$, such that the cardinality of $B_{\alpha\lambda}$ is $c_{\alpha\lambda}$. Now $MF(\bigvee_\gamma b_{\alpha\lambda\gamma}) = \{(e_\lambda, c_{\alpha\lambda})\} = MF(\bigvee_\gamma a_{\alpha\lambda\gamma})$. That is, $\bigvee_\gamma b_{\alpha\lambda\gamma} \sim \bigvee_\gamma a_{\alpha\lambda\gamma}$. For each $\alpha \in I$, define $b_\alpha = \bigvee_\lambda (\bigvee_\gamma b_{\alpha\lambda\gamma})$. Note that $b_\alpha \wedge e_{\lambda_o} = e_{\lambda_o} \wedge \bigvee_\lambda (\bigvee_\gamma b_{\alpha\lambda\gamma}) = \bigvee_\lambda (e_{\lambda_o} \wedge \bigvee_\gamma b_{\alpha\lambda\gamma})$
$= \bigvee_\gamma b_{\alpha\lambda_o\gamma} \sim \bigvee_\gamma a_{\alpha\lambda_o\gamma} = a_\alpha \wedge e_{\lambda_o}$ for all λ_o. By lemma 4.6.

$b_\alpha = b_\alpha \wedge \bigvee_\lambda e_\lambda = \bigvee_\lambda (b_\alpha \wedge e_\lambda) \sim \bigvee_\lambda (a_\alpha \wedge e_\lambda) = a_\alpha \wedge \bigvee_\lambda e_\lambda = a_\alpha$. Furthermore, $b = b \wedge \bigvee_\lambda e_\lambda = \bigvee_\lambda (b \wedge e_\lambda) = \bigvee_\lambda (\bigvee_\alpha \bigvee_\gamma b_{\alpha\lambda\gamma}) = \bigvee_\alpha (\bigvee_\lambda \bigvee_\gamma b_{\alpha\lambda\gamma}) = \bigvee_\alpha b_\alpha$.

Discussion 4.9. If $(L,\sim)$ is a Loomis dimension lattice, we define a certain set of elements called $\sim$-invariant elements by the rule that e is $\sim$-invariant if and only if no subelement of e is equivalent to any subelement of e'. The set of all $\sim$-invariant elements of L form a subcomplete, suborthomodular sublattice of the center of L and, hence, determines a center valued quantifier $\phi_\sim$ on L. If the equivalence relation $\sim$ was defined from some quantifier ϕ, as in definition 4.1, it follows from Loomis' work [9] that $\phi = \phi_\sim$. On the other hand, if $(L,\sim)$ satisfies condition B', and if we define an equivalence relation as in definition 4.1 using the quantifier $\phi_\sim$, we will obtain the original equivalence relation. Loomis' work implies that there is a one to one correspondence between center valued quantifiers and dimension equivalence relations, provided that the quantifier was originally determined by a dimension equivalence relation. The question, which is partially solved in this paper, then arises, how can one reasonably characterize monadic lattices which give rise to dimension lattices? Lemmas 4.2, 4.3, 4.7, and 4.8 together with Loomis' work show that these lattices may be characterized in the atomic case by piecewise semimodularity. We now formalize this discussion in the following theorem.

Theorem 4.10. (i) If (L,ϕ) is a type I monadic lattice with a center valued quantifier which determines a Loomis dimension lattice in the manner of definition 4.1, then (L,ϕ) is piecewise semimodular.

(ii) If (L,ϕ) is an atomic, piecewise semimodular, monadic lattice with a center valued quantifier and if $(L,\sim)$ is defined as in definition 4.1, then $(L,\sim)$ is a completely divisible Loomis dimension lattice.

(iii) There is a one to one correspondence between completely divisible, atomic Loomis dimension lattices and piecewise semimodular, atomic, monadic lattices with center valued quantifiers.

Proof: (i) follows from Loomis' work. In (ii), $\sim$ is clearly an equivalence relation. The rest follows from lemmas 4.2, 4.3, 4.7, and 4.8. (iii) follows from (ii) and Loomis' work.

CHAPTER V

CONCLUSION

Piecewise semimodularity seems to be only a stepping stone to characterizing monadic lattices which may be equipped with a dimension equivalence relation. The following discussion suggests that there might be some kind of "refinement" condition equivalent to piecewise semimodularity.

Let a be an element of a monadic lattice $(L,|\ |)$, with a center valued quantifier. Also, let L be atomic and $|a|$ be an atom in the range of the quantifier. Suppose that there are two, finite, orthogonal families of nonzero, simple elements $\{b_i: i = 1, \ldots, n\}$ and $\{c_j: j = 1, \ldots, m\}$, such that $|b_i| = |c_j| = |a|$ and $\bigvee_i b_i = \bigvee_j c_j = a$. As in Chapter III, it follows that the elements b_i and c_j are atoms in L. Associated with the two, orthogonal families above are the two chains $\{\bigvee_{i=1}^{k} b_i: k = 1, \ldots, n\}$ and $\{\bigvee_{j=1}^{p} c_j: p = 1, \ldots, m\}$, respectively. If $n < m$, a refinement condition would assert that a nonzero element d could be inserted properly in the shorter chain, that is, for some k, $\bigvee_{i=1}^{k} b_i < d < \bigvee_{i=1}^{k+1} b_i$. Now $d = d \wedge \bigvee_{i=1}^{k+1} b_i = d \wedge (b_{k+1} \vee \bigvee_{i=1}^{k} b_i) = (d \wedge b_{k+1}) \vee \bigvee_{i=1}^{k} b_i$. Since b_{k+1} is an atom, $d \wedge b_{k+1} = 0$ or b_{k+1}. Thus, $d = \bigvee_{i=1}^{k} b_i$ or $\bigvee_{i=1}^{k+1} b_i$, a contradiction. Thus a refinement condition would imply that $m = n$, which is the essence of piecewise semimodularity in this situation.

The study of the relationship between dimension lattices and monadic lattices is only one part of a much larger problem. It has been observed that there is an intimate connection between the following four distinct mathematical structures: (1) completely divisible dimension lattices $(L,\sim)$, (2) monadic lattices with center valued quantifiers (L,δ), (3) Baer *-semigroups with some form of *-equivalence $(S,\underset{*}{\sim})$, and (4) abstract spectral measures (L,X,μ). The connections between these structures are illustrated in Figure 2. The solid lines indicate complete results, and the dashed lines indicate partial results. Connection (i) is due to L. H. Loomis, and is found in

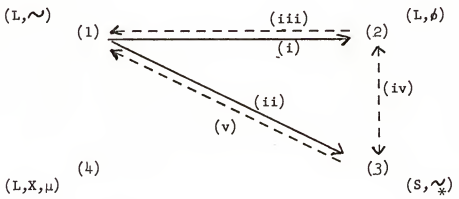

Figure 2. The Quadripartite Diagram

American Mathematical Society Memoir 18 [9]. Connection (ii) is due to M. F. Janowitz [7]. Connection (iii) is given in theorem 4.10. David Foulis [2] has given a one to one correspondence between orthomodular lattices and Baer *-semigroups, and M. F. Janowitz [7] has shown how a center valued quantifier on an orthomodular lattice is carried over to the coordinatizing Baer *-semigroup. However, in (iv), one still needs, among other things, to see how piecewise semimodularity affects the

coordinatizing Baer *-semigroup. In (v), there have been several partial results, including work by David Foulis [1] and Irving Kaplansky [8]. There has been very little done concerning generalized spectral measures, except in the case of $\overline{L(H)}$ where H is a Hilbert space. These results may be found in Introduction to Hilbert Space and the Theory of Spectral Multiplicities by P. R. Halmos [3]. Also see [4], [5], and [10].

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BIOGRAPHICAL SKETCH

Jean Harwell Bevis was born on December 6, 1939, at Miami, Florida. He attended public schools there and was graduated from Coral Gables High School in 1957. His undergraduate studies were undertaken at the University of Florida in September of 1957. In January, 1961, he received the degree of Bachelor of Science with honors. In February, 1961, he enrolled in the Graduate School of the University of Florida. Since entering Graduate School he has worked as a graduate assistant and as an interim instructor in the Department of Mathematics, teaching various undergraduate courses in mathematics. In June, 1962, he received the degree of Master of Science. He is a member of the Mathematical Association of America and the American Mathematical Society. He is married to the former Mary Patricia Cobb, and he has a daughter named Sharon Elaine.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 14, 1965

Ernest H. Cox
Dean, College of
Arts and Sciences

Dean, Graduate School

Supervisory Committee:

D. J. Forbes
Chairman

Ryder Pinder

J. E. Brinnam